

Section 12.3

Solving Trigonometric Equations

Equations involving the trigonometric functions may be solved using any of the usual equation solving techniques such as factoring, quadratic formula, or graphing calculator. Generally you should attempt to isolate each trigonometric function and set it equal to a number. Then determine which angles satisfy that equation. Since there is usually more than one angle, be sure to find them all. Often the domain will be given in the equation. If so, be sure to find all the solutions in the domain.

Example 1: Solve $\cos x = -\frac{1}{2}$ for $0 \leq x \leq 2\pi$.

Solution: The reference angle for all the solutions is $\frac{\pi}{3}$ since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. The angles are found in quadrants II and III where cosines are negative.

$$\text{Solution set} = \left\{ \frac{2\pi}{3}, \frac{4\pi}{3} \right\}.$$

Example 2: Solve $\tan x = -1.2453$ for $0^\circ \leq x \leq 360^\circ$. Round your answer to the *nearest three decimal places*.



Solution: Remember to set your calculator to degree mode. Use the \tan^{-1} button to find the reference angle. Enter $\tan^{-1}(1.2453)$. This gives 51.23486153° . Since tangents are negative in quadrants II and IV the solutions are:

$$180^\circ - 51.23486153^\circ = 128.765^\circ$$

$$360^\circ - 51.23486153^\circ = 308.765^\circ$$

A calculator solution is shown below. Notice how the reference angle is immediately stored to a variable, A, and then quickly recalled to get the two answers. This is a technique that saves typing and, more importantly, avoids the possibility of making a mistake in entering a long decimal or rounding off too soon. Also notice that, as always, the rounding is done last.

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tan-1(1.2453)→A
51.23486153
180-A
128.7651385
360-A
308.7651385
■
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The following examples show how different equation solving methods are applied to equations with trigonometric functions.

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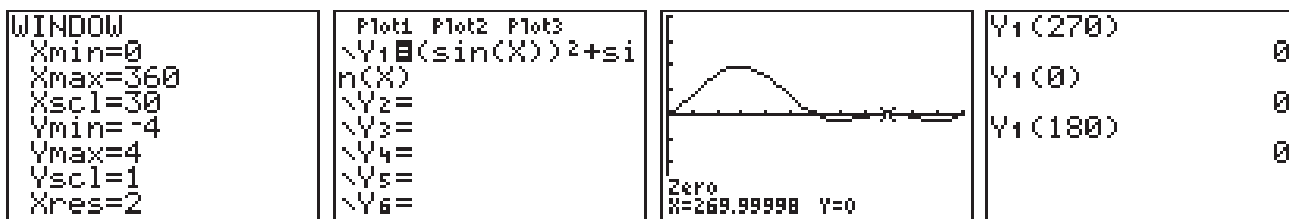
Example 3: Solve $\sin^2 x + \sin x = 0$ for all x such that $0^\circ \leq x \leq 360^\circ$.

Algebraic solution: Begin by factoring:

$$\begin{aligned} \sin x(\sin x + 1) &= 0 \\ \sin x &= 0; x = 0^\circ, 180^\circ, 360^\circ \\ \text{or} \\ \sin x &= -1; x = 270^\circ \end{aligned}$$



Calculator Solution: Graph the left side of the equation in a window that includes the given domain. Trace or use the built-in zero finder to find the roots. The roots must be found one at a time. Notice that the zero-finder is not always exact. You should suspect that the answer is actually $x = 270^\circ$ and check this value. Intelligent guessing is also allowed. From the graph it appears as if $x = 0^\circ$, $x = 360^\circ$, and even $x = 180^\circ$ are roots. Checking these values may be faster than using the zero-finder.



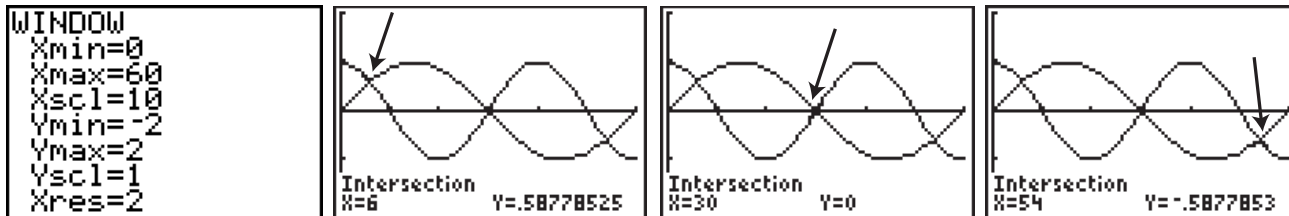
MODEL PROBLEM: (From the January 2003 B Exam #20)

If $\sin 6A = \cos 9A$, then the value of A is

- (1) 6
- (2) 36
- (3) 54
- (4) $1\frac{1}{2}$



Graphic Solution: Graph the functions $Y_1 = \sin 6x$ and $Y_2 = \cos 9x$ in a window wide enough to cover the answer choices. Since the answers appear to be degrees, make sure your calculator is in degree MODE.



There are intersections in the interval (and a lot more between 0° and 360°). Two of them are among the answer choices. Both choices (1) 6 and (3) 54 were counted as correct.

Algebraic Solution: This type of problem was often asked on the Regents Exams given before the Math B Exam was introduced and before graphing calculators were required. The intended solution is

$$\begin{aligned}\sin 6A &= \cos 9A \\ 6A + 9A &= 90^\circ \\ 15A &= 90^\circ \\ A &= 6^\circ\end{aligned}$$

The idea being tested here is that if two angles, A and B , are complementary then any trig function of one is equal to the cofunction of the other ($\sin A = \cos B$, $\tan A = \cot B$, and $\sec A = \csc B$). While that statement is true, its converse is not. The fact that the sine of one angle is equal to the cosine of another does not necessarily mean the angles are complementary. As the graphic solution shows, there are many other solutions. Therefore, graphing is the safest way to approach this problem. If you are given that $6A$ and $9A$ are both acute angles and $\sin 6A = \cos 9A$, only then will the algebraic approach give the correct answer.

Example 4: Solve to the nearest thousandth $\sin^2 x + 3 \sin x = -1$ for all x such that $0^\circ \leq x \leq 360^\circ$.

Solution: The equation is a quadratic in terms of $\sin x$. Begin by setting it equal to zero. The resulting equation does not factor so use the quadratic formula.

$$\begin{aligned}\sin^2 x + 3 \sin x + 1 &= 0 \\ \sin x &= \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2} = \frac{-3 \pm \sqrt{5}}{2}\end{aligned}$$

$$\sin x \approx -0.381996; x = 202.456^\circ \text{ or } x = 337.544^\circ$$

$$\sin x \approx -2.6180; \text{ therefore no additional value of } x \text{ exists.}$$

The angles in the third line above were found by evaluating $\sin^{-1} \left(\left| \frac{-3 + \sqrt{5}}{2} \right| \right)$ on a calculator to find the reference angle, as in Example 2. The number $\frac{(-3 - \sqrt{5})}{2}$ is less than -1 and is not in the domain of the sine function.

The solution set is $\{202.456^\circ, 337.544^\circ\}$.

Example 5: Solve the equation: $\frac{1 - \sin \beta}{\cos \beta} = \frac{\cos \beta}{1 + \sin \beta}$ for all β , $0^\circ \leq \beta \leq 360^\circ$.

Solution: Begin by multiplying both sides by the common denominator. This gives an equation with a denominator of one.

$$\begin{aligned}\frac{1 - \sin \beta}{\cos \beta} \cdot (1 + \sin \beta)(\cos \beta) &= \frac{\cos \beta}{1 + \sin \beta} \cdot (1 + \sin \beta)(\cos \beta) \\ (1 - \sin \beta)(1 + \sin \beta) &= (\cos \beta)(\cos \beta) \\ 1 - \sin^2 \beta &= \cos^2 \beta \\ \cos^2 \beta &= \cos^2 \beta \\ 0 &= 0\end{aligned}$$

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The last three lines are identities. Therefore the equation is true for all β in its domain. The domain does not include the values that make the denominators zero. The solution is all numbers from zero degrees to 360° inclusive, except 90° and 270° .



If you solved this equation by graphing both sides separately and finding the points of intersection, both graphs appear to be identical – that is, they intersect everywhere! The two values that are not in the domain probably would not be obvious from a calculator graph; these must still be considered separately.

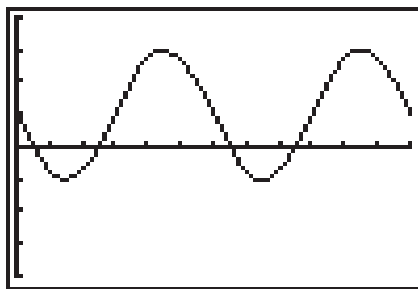
Example 6: Solve the equation $1 - 2 \sin(2x) = 0$ for all values of x , $0^\circ \leq x \leq 360^\circ$.

Solution: Begin by solving for $2x$. Since solving for x requires dividing by 2, you must consider values of x from 0° to 720° . Those larger than 360° , when divided by 2, will become less than 360° .

$$\begin{aligned} 1 - 2 \sin(2x) &= 0 \\ -2 \sin(2x) &= -1 \\ \sin(2x) &= \frac{1}{2} \\ 2x &= 30^\circ, 150^\circ, 390^\circ, 510^\circ \\ x &= 15^\circ, 75^\circ, 195^\circ, 255^\circ \end{aligned}$$



You may verify the answers by graphing in a window with $0^\circ \leq x \leq 360^\circ$. The tick marks in the figure are at 30° intervals. There are 4 solutions.



Example 7: Solve $\sin\left(x + \frac{\pi}{2}\right) = \tan x$ for all values of x , $0 \leq x \leq 2\pi$. Give your answers correct to three decimal places.

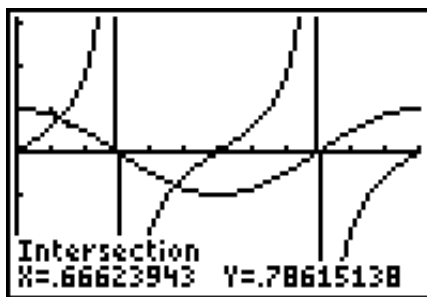
Solution:

$$\begin{aligned} \sin\left(x + \frac{\pi}{2}\right) &= \tan x \\ \sin(x) \cos\left(\frac{\pi}{2}\right) + \cos(x) \sin\left(\frac{\pi}{2}\right) &= \tan x \\ \sin(x)(0) + \cos(x)(1) &= \frac{\sin x}{\cos x} \\ \cos x &= \frac{\sin x}{\cos x} \\ \cos^2 x &= \sin x \\ 1 - \sin^2 x &= \sin x \\ 0 &= \sin^2 x + \sin x - 1 \\ \sin x &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2} \\ \sin x &\approx 0.618034; x = 0.666, 2.475 \\ &\text{or} \\ \sin x &\approx -1.618034 < -1; \text{ therefore no additional value of } x \text{ exists} \end{aligned}$$

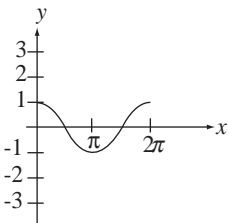
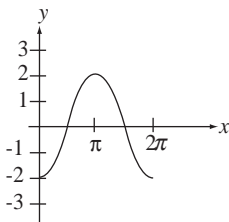
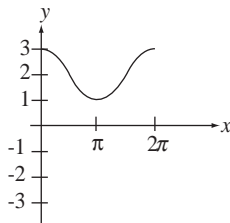
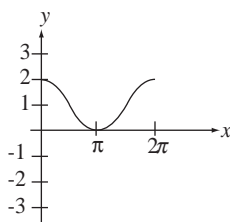
The solution is $x = 0.666$ radians or $x = 2.475$ radians.



Graphing both sides of the equation separately and finding the x -coordinates of their points of intersection is another way to solve this problem.



Part I Questions

1. When the following pairs of functions are graphed, which pair could *not* be used to solve the equation $3 \cos x + 2 = 0$?
- (1) $y = \cos x$ and $y = -2$
 - (2) $y = \cos x$ and $y = -\frac{2}{3}$
 - (3) $y = 3 \cos x$ and $y = -2$
 - (4) $y = 3 \cos x + 2$ and $y = 0$
2. How many solutions are there to the equation $\cos^2 x = \cos x$ if $0 \leq x \leq 2\pi$?
- (1) 1
 - (2) 2
 - (3) 3
 - (4) 4
3. What are all values of x such that $0 \leq x \leq 2\pi$ and $\cos^2 x - \sin^2 x = 0$?
- (1) $\{0, \pi, 2\pi\}$
 - (2) $\{\frac{\pi}{2}, \frac{3\pi}{2}\}$
 - (3) $\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$
 - (4) $\{\frac{\pi}{4}, \frac{3\pi}{4}\}$
4. If the equation $A \cos x + C = 0$ has no solution in the interval $0 \leq x \leq 2\pi$, then the graph of $y = A \cos x + C$ could be
- (1) 
 - (2) 
 - (3) 
 - (4) 
5. If $3 \cos x - \sec x = 0$ and $90^\circ \leq x \leq 180^\circ$, then $x \approx$
- (1) 54.736
 - (2) 125.264
 - (3) 137.264
 - (4) 144.736
6. For what acute angle β is $\cos \beta = \sin(46^\circ)$?
- (1) 23°
 - (2) 44°
 - (3) 46°
 - (4) 54°
7. If $6A$ and $9A$ are both acute angles, and $\tan(6A) = \cot(9A)$, then the value of A in degrees is
- (1) 6
 - (2) 36
 - (3) 54
 - (4) $1\frac{1}{2}$
8. If A and B are acute angles measured in degrees, and $\sin A = \cos B$, which statement is true?
- (1) $A = B$
 - (2) $A = 90 - B$
 - (3) $A = 180 - B$
 - (4) $A = B + 45$
9. If A and B are both between 0 and $\frac{\pi}{2}$, and $\cos B = \sin A$, which statement is always true?
- (1) $A = B$
 - (2) $A + B = \frac{\pi}{2}$
 - (3) $A + B = \pi$
 - (4) $B = \frac{\pi}{4} + A$

Part II Questions

10. What are all values of x , $0 \leq x \leq 2\pi$, for which $2 \sin^2 x + \sin x = 1$?
11. To the *nearest hundredth of a radian*, find all values of x in the interval $0 < x < 2\pi$ which satisfy the equation $\cot x = -6.38$.
12. Solve the equation $\sin(2\theta) = \frac{1}{2}$ for all values of θ such that $0 < \theta < 360^\circ$.
13. Solve the equation $\cos\left(\frac{1}{2}\theta\right) = \frac{1}{2}$ for all θ such that $0 \leq \theta \leq 2\pi$.
14. Solve $4 \sin x - 3 \csc x = 0$ for all values of x such that $0^\circ \leq x \leq 360^\circ$.
15. Solve $2 \sin \theta - 1 = \csc \theta$ for all values of θ such that $0^\circ \leq \theta \leq 360^\circ$.
16. Solve $\sin(2x) = \cos(x)$ for all x such that $0 \leq x \leq 2\pi$.
17. For how many values of x in the interval $0^\circ \leq x \leq 360^\circ$ does $\sin(2x) = \cos(3x)$? Explain your reasoning.
18. What are the values of x , $0 \leq x \leq 2\pi$, for which $(2 \cos x - 1)(4 \cos^2 x - 3) = 0$?
19. The maximum height, h , reached by a projectile, can be modeled by the equation $h = \frac{v^2 \sin^2 \theta}{64}$ where v is the initial velocity in feet per second and θ is the measure of the angle at which the projectile is launched. A football is kicked with an initial velocity of 70 feet per second. At what acute angle, to the *nearest degree*, should it be struck to attain a maximum height of 40 feet?

Part III Questions

20. Solve $\cos^2 \theta + \sin \theta + 1 = 0$ for $0^\circ \leq \theta < 360^\circ$.
21. If $f(x) = \sin x$ and $g(x) = 2x$, for how many values of x , $0 \leq x \leq 2\pi$, does $f(g(x)) = g(f(x))$?
22. The horizontal distance, d , traveled by a projectile, can be modeled by the equation $d = \frac{v^2}{32} \sin 2\theta$ where v is the initial velocity in feet per second and θ is the measure of the angle at which the projectile is launched. A golf ball is hit with an initial velocity of 110 feet per second.
- a At what angle(s), to the *nearest hundredth*, should it be struck to travel 350 feet?
- b For any given initial velocity, what angle will maximize the horizontal distance traveled?
23. The height, h , in feet above the ground, of a section of a roller coaster can be modeled by the function $h(t) = 150 + 150 \cos\left(\frac{\pi}{75}t\right)$, $0 \leq t \leq 150$, where t is the horizontal distance in feet from the top of the highest point on the ride.
- a What is the horizontal distance in feet between the two high points on the ride?
- b For what values of t is the roller coaster 75 feet above the ground?

Part IV Questions

24. On the accompanying grid, graph and label $y = 3 \cos x + 1$ and $y = \cos(2x)$ for $0 \leq x \leq 2\pi$. Determine *algebraically* the coordinates of the points where the graphs intersect.
25. On the accompanying grid, graph and label $y = -3 \sin(x) - 1$ and $y = \cos(2x)$, for $0 \leq x \leq 2\pi$. Determine *algebraically* the coordinates of the points of intersection of the graphs.

[Use graph paper for this question.]

[Use graph paper for this question.]